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RANKING EXTREME EFFICIENT DECISION MAKING UNITS IN STOCHASTIC DEA

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Abstract. We develop L_1 -norm model assuming that is always feasible and bounded for ranking extreme efficient decision making units (DMUs) in stochastic data envelopment analysis (DEA). And also, we present a deterministic equivalent of stochastic model. It is shown that this deterministic model can be convert to a quadratic program. Finally, the proposed model has been implemented for ranking efficient units of 30 universities in IRAN that we intend to evaluate the universities from educational evaluation.

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1 Introduction

In order to incorporate stochastic input and output variations into the DEA analysis Segupta (1982) generalized the CCR ratio model by defining measure of the relative efficiency of a DMU as the maximum of the sum of the expected ratio of weighted outputs to weighted inputs and a reliability function subject to several chance constraints. Cooper et al. (1996), incorporated the satisfying concepts of Simon into DEA and developed the satisfying DEA model. More recently, stochastic input and output variations into DEA have been studied by Asgharian et al. (2010); Khodabakhshi (2009) and Khodabakhshi & Asgharian (2009).

Ranking of efficient DMUs is very important question and many DEA researchers and practitioners have studied about it. Anderson & Peterson (1993) were first addressed this question in their seminal paper where they introduced super-efficiency models to rank efficient decision making units. Aboud & Nachaoui (2020) are considered the equilibrium problem associated to semiconductor. This is the case where no tension is applied on the contacts of the device. The problem is discretized using finite difference Methods. In order to solve the resulting discrete problem, a single rank quasi Newton method is introduced to make the solution of the original nonlinear problem easier. We compared this method with other classical methods. Also, Jahanshahloo et al. (2004) introduced L_1-norm model for ranking of efficient DMUs.

In this paper, we will extend Stochastic L_1-norm model, allowing deterministic inputs and outputs to be stochastic. Then, we obtain a deterministic equivalent to our stochastic model and show this deterministic equivalent can be transformed to a quadratic programming model.

2 Background

We consider n homogeneous DMUs $\{DMU_j|$j=1,...,n$\}$ each having m inputs denoted by $x_j \in R^m(j = 1, \ldots, n)$ and s outputs denoted by $y_j \in R^s$(j = 1, ..., n). We assume that x_j and y_j are non-negative deterministic elements. The production possibility set (PPS) is defined as follows

$$T_c = \left\{ (x,y) \middle| (x,y) \sum_{j=1}^{n} \lambda_j \ x_j \le x, \ \sum_{j=1}^{n} \lambda_j \ y_j \ge y, \ \lambda_j \ge 0, j = 1, \ldots, n \right\}.$$

Assume that DMU_0 is one of the extreme efficient DMUs. By omitting DMU_0 from T_c, we define the production possibility set $T_c{}'$ as

$$T_c{}' = \left\{ (x,y) \middle| (x,y) \sum_{j=1, j\neq 0}^{n} \lambda_j \ x_j \le x, \ \sum_{j=1, j\neq 0}^{n} \lambda_j \ y_j \ge y, \ \lambda_j \ge 0, j = 1, \ldots, n \right\}.$$

$L_1 - norm$ model is one of the important models for ranking of efficient DMUs. This model was introduced in Anderson & Peterson (1993). It is used to rank DM U_0 in below

$$\min \mu_c{}^0(X,Y) = \sum_{i=1}^{m} |x_i \ - x_{i0}| + \sum_{r=1}^{s} |y_r - y_{r0}|, \tag{1}$$

s.t.

$$\sum_{\substack{j=1 \\ j\neq 0}}^{n} \lambda_j \ x_{ij} \le \ x_i,$$

$$\sum_{\substack{j=1 \\ j\neq 0}}^{n} \lambda_j \ y_{rj} \ge \ y_r \ ,$$

$$x_i \ge 0 \qquad , i = 1, \ldots, m,$$

$$y_r \ge 0 \qquad , r = 1, \ldots, s,$$

$$\lambda_j \ge 0 \qquad , j = 1, \ldots, n.$$

It was proved that model (1) was converted to a linear program (Jahanshahloo et al., 2004) as:

$$\min \mu_c{}^0(X,Y) = \sum_{i=1}^{m} x_i - \sum_{r=1}^{s} y_r + A, \tag{2}$$

s.t.

$$\sum_{\substack{j=1 \\ j\neq 0}}^{n} \lambda_j \ x_{ij} \le \ x_i,$$

$$\sum_{\substack{j=1 \\ j\neq 0}}^{n} \lambda_j \ y_{rj} \ge \ y_r \ ,$$

$$x_i \geq x_{i0},$$
$$y_r \leq y_{r0},$$
$$x_i \geq 0 \ , \quad i = 1, \ldots, m,$$
$$y_r \geq 0, \quad r = 1, \ldots, s,$$
$$\lambda_j \geq 0, \qquad\qquad j = 1, \ldots, n,$$

where $A = -\sum_{i=1}^{m} x_{i0} + \sum_{r=1}^{s} y_{r0}$.
Theorem 1: Model (2) is always feasible and bounded.
Proof: Refer to Jahanshahloo et al. (2004).

3 Stochastic L_1-norm

We are going to develop the L_1-norm model (Jahanshahloo et al., 2004) in stochastic data envelopment analysis to rank extreme efficient DMUs. Following Cooper et al. (2004), let $\widetilde{x}_j = (\widetilde{x}_{1j}, ..., \widetilde{x}_{mj})$, $\widetilde{y}_j = (\widetilde{y}_{1j}, ..., \widetilde{y}_{sj})$ be random input and output related to DM$U_j (j = 1, ..., n)$. Let also $\overline{x}_j = (\overline{x}_{1j}, ..., \overline{x}_{mj})$, $\overline{y}_j = (\overline{y}_{1j}, ..., \overline{y}_{sj})$ show the corresponding vectors of expected values of inputs and outputs for DMU_j. Suppose that all input and output components are jointly normally distributed in the following chance constrained version of the stochastic model (2) with inequality constraints

$$\min {R_c}^0(X, Y) = E(\sum_{i=1}^{m} \widetilde{x}_i - \sum_{r=1}^{s} \widetilde{y}_r + A), \tag{3}$$

s.t.

$$\sum_{\substack{j=1 \\ j \neq 0}}^{n} P(\lambda_j \ \widetilde{x}_{ij} \leq \ \widetilde{x}_i) \geq 1 - \alpha,$$
$$\sum_{\substack{j=1 \\ j \neq 0}}^{n} P(\lambda_j \ \widetilde{y}_{rj} \geq \ \widetilde{y}_r) \geq 1 - \alpha,$$
$$P(\widetilde{x}_i \geq \widetilde{x}_{i0}) \geq 1 - \alpha,$$
$$P(\widetilde{y}_r \leq \widetilde{y}_{r0}) \geq 1 - \alpha,$$
$$\widetilde{x}_i \geq 0 \ , \quad i = 1, \ldots, m,$$
$$\widetilde{y}_r \geq 0, \quad r = 1, \ldots, s,$$
$$\lambda_j \geq 0, \qquad\qquad j = 1, \ldots, n.$$

where α is a predetermined value between 0 and 1, represents the probability measure. The corresponding stochastic version of model (3), including slack variables is as follows

$$\min {R_c}^0(X, Y) = E(\sum_{i=1}^{m} \widetilde{x}_i) - E(\sum_{r=1}^{s} \widetilde{y}_r) + A, \tag{4}$$

s.t.

$$\sum_{\substack{j=1 \\ j \neq 0}}^{n} P(\lambda_j \ \widetilde{x}_{ij} + s_i^- \leq \ \widetilde{x}_i) = 1 - \alpha,$$

$$\sum_{\substack{j=1\\ j\neq 0}}^{n} P(\lambda_j\ \widetilde{y}_{rj} - s_r^+ \geq\ \widetilde{y}_r) = 1-\alpha,$$

$$P(\widetilde{x}_i - t_i^+ \geq \widetilde{x}_{i0}) = 1-\alpha,$$

$$P(\widetilde{y}_r + t_r^- \leq \widetilde{y}_{r0}) = 1-\alpha,$$

$$\widetilde{x}_i \geq 0\ ,\quad s_i^- \geq 0,\quad t_i^+ \geq 0,\quad i=1,\ldots,m,$$

$$\widetilde{y}_r \geq 0,\quad s_r^+ \geq 0,\quad t_i^- \geq 0,\quad r=1,\ldots,s,$$

$$\lambda_j \geq 0,\qquad j=1,\ldots,n.$$

4 Deterministic equivalent

In this section, we exploit the normality assumption to introduce deterministic equivalent to the model (4), that is:

$$\min R_c{}^0(X,Y) = E(\sum_{i=1}^{m} \overline{x}_i) - E(\sum_{r=1}^{s} \overline{y}_r) + A, \tag{5}$$

s.t.

$$\sum_{\substack{j=1\\ j\neq 0}}^{n} \lambda_j\ \overline{x}_{ij} + s_i^- + \sigma_i^0(\lambda)\varphi^{-1}(\alpha) = \overline{x}_i,$$

$$\sum_{\substack{j=1\\ j\neq 0}}^{n} \lambda_j\ \overline{y}_{rj} - s_r^+ + \sigma_i^1(\lambda)\varphi^{-1}(\alpha) = \overline{y}_r,$$

$$\overline{x}_i - t_i^+ + var(x_i)\varphi^{-1}(\alpha) = x_{i0},$$

$$\overline{y}_r + t_r^- + var(y_r)\varphi^{-1}(\alpha) = y_{r0},$$

$$\overline{x}_i \geq 0\ ,\quad s_i^- \geq 0,\quad t_i^+ \geq 0,\quad i=1,\ldots,m,$$

$$\overline{y}_r \geq 0,\quad s_r^+ \geq 0,\quad t_r^- \geq 0,\quad r=1,\ldots,s,$$

$$\lambda_j \geq 0,\qquad j=1,\ldots,n,\ j\neq 0,$$

where φ is the cumulative distribution function of standard normal random variable and φ^{-1} is its inverse, we assume that $\overline{x}_{ij}$ and $\overline{y}_{ij}$ are the means of the input and output variables, which can be estimated by the observed values of the inputs and outputs using the aforementioned property of normal distribution, one can show that

$$(\sigma_i^0(\lambda))^2 = \sum_{j=1,j\neq 0}^{n}\ \sum_{k=1,k\neq 0}^{n} \lambda_j\lambda_k cov(\widetilde{x}_{ij},\widetilde{x}_{ik}) + 2(\lambda_0-1) \sum_{j=1,j\neq 0}^{n} \lambda_j cov(\widetilde{x}_{ij},\widetilde{x}_{i0}) + (\lambda_0-1)^2 var(\widetilde{x}_{i0}),$$

$$(\sigma_i^1(\lambda))^2 = \sum_{k=1,k\neq 0}^{n}\ \sum_{j=1,j\neq 0}^{n} \lambda_k\lambda_j cov(\widetilde{y}_{rk},\widetilde{x}_{rj}) + 2(\lambda_0-1) \sum_{k=1,k\neq 0}^{n} \lambda_j cov(\widetilde{y}_{rk},\widetilde{y}_{r0}) + (\lambda_0-1)^2 var(\widetilde{y}_{r0}).$$

5 Application

Consider 30 universities in IRAN and assume that we intend to evaluate the universities from educational evaluation. For education evaluation, for each university consider three inputs: university score, number of faculty members, and number of students, and two outputs: income and number of students ready to defend their theses. all input and output data have stochastic values with normal distribution so the average an variance of them are shown in Table1 as (average,variance). The efficiencies of all DMUs have been computed by using the stochastic-BCC model, then for ranking efficient DMUSs have been used the stochastic L_1-norm model. The results are provided in Table1.

6 Conclusion

Stochastic models may be better suited for DEA when there is uncertainty associated with the inputs and/or outputs of DMUs or when an analyst may be wondering how much change can be incurred in the ranking of DMUs if sum of the inputs and/or outputs change. In this paper, we have developed Stochastic L_1-norm model for ranking extreme efficient DMUs in stochastic data envelopment analysis. And We have obtained the deterministic equivalent for the stochastic version. Finally, the proposed model has been implemented for ranking efficient units of 30 universities in IRAN that we intend to evaluate the universities from educational evaluation. Applying the proposed approach in different norms, practically, would be interesting for further research.

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Table 1: Inputs and outputs of educational evaluation

DMU	*input1*	*input2*	*input3*	*output1*	*output2*	*Stoch − BBC*	*Stoch − L1 − Norm*
1	(73504.25, 400000)	(86, 4.00)	(4000, 63333)	(57029.56, 113200)	(40, 6.33)	0.97	
2	(36582.75, 1100000)	(89, 4.33)	(2565, 6908)	(36872.00, 491600)	(18, 1.00)	0.92	
3	(25002.25, 700000)	(87, 12.33)	(1343, 2653)	(38680.00, 10000)	(20, 16.33)	1.00	5
4	(36684.75, 200000)	(94, 4.33)	(1500, 2500)	(35933.00, 351500)	(90, 13.00)	0.92	
5	(36834.50, 1300000)	(97, 6.33)	(1680, 4233)	(54457.78, 46800)	(30, 17.33)	1.00	2
6	(62611.63, 2100000)	(90, 3.00)	(3750, 5833)	(72277.11, 309600)	(11, 12.33)	1.00	3
7	(41572.77, 1200000)	(89, 4.00)	(3313, 3206)	(36625.44, 208800)	(101, 10.33)	0.89	
8	(38851.85, 200)	(89, 24.33)	(1500, 2500)	(46360.33, 209200)	(48, 16.33)	0.96	
9	(95522.75, 800000)	(91, 0.33)	(1600, 2500)	(86063.33, 377200)	(71, 6.33)	1.00	7
10	(39697.81, 500000)	(78, 1.00)	(1725, 1458)	(47242.11, 31900)	(34, 4.00)	0.92	
11	(40736.13, 100000)	(89, 2.00)	(1920, 1300)	(38977.78, 358000)	(184, 50.33)	1.00	1
12	(27300.63, 1500000)	(93, 1.33)	(1408, 2921)	(38214.89, 156900)	(28, 4.33)	0.98	
13	(53669.46, 300000)	(93, 1.33)	(2500, 3333)	(58340.44, 164700)	(45, 2.33)	0.90	
14	(94969.88, 400000)	(93, 3.00)	(2800, 10833)	(88472.33, 32000)	(40, 6.33)	1.00	8
15	(50062.00, 300000)	(85, 6.33)	(1630, 4300)	(50499.00, 385300)	(14, 70.33)	0.91	
16	(45926.25, 200)	(94, 4.00)	(1127, 4663)	(47907.22, 2900)	(25, 8.33)	1.00	12
17	(66259.14, 200000)	(92, 6.33)	(1477, 13326)	(59579.78, 26700)	(74, 16.00)	0.94	
18	(88782.50, 100000)	(85, 6.33)	(1304, 5905)	(83075.11, 85400)	(23, 16.33)	1.00	6
19	(35453.47, 300000)	(100, 13.00)	(1621, 1330)	(51026.56, 328200)	(32, 13.00)	1.00	9
20	(33196.50, 300)	(89, 2.33)	(1340, 1808)	(29658.11, 243200)	(77, 19.00)	0.90	
21	(8402.75, 19554)	(121, 10.33)	(1393, 2793)	(27735.00, 7900)	(19, 20.33)	1.00	10
22	(122897.88, 40766)	(100, 6.33)	(2191, 13030)	(102855.11, 84800)	(47, 36.33)	1.00	4
23	(32587.75, 100)	(92, 9.33)	(140, 5952)	(34063.67, 212200)	(23, 6.33)	1.00	12
24	(60866.38, 200000)	(90, 13.00)	(1231, 4687)	(53731.33, 1050100)	(63, 16.33)	0.98	
25	(84279.99, 300000)	(100, 6.33)	(1907, 2886)	(75776.56, 2200)	(77, 13.00)	0.99	
26	(33196.50, 120)	(100, 30.33)	(1340, 6916)	(29658.11, 43200)	(77, 12.33)	0.90	
27	(33196.50, 600000)	(93, 1.00)	(1340, 608)	(29658.11, 20200)	(77, 74.33)	0.91	
28	(66803.13, 300)	(82, 0.33)	(1603, 11186)	(72552.67, 11600)	(75, 25.00)	1.00	11
29	(40156.13, 300000)	(100, 2.00)	(2300, 5200)	(38630.78, 23400)	(24, 4.00)	0.96	
30	(33196.50, 200000)	(83, 16.33)	(1340, 2233)	(29658.11, 11200)	(77, 110.33)	0.89	